# The Limit Equilibrium Methods

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EPFL / ENAC / GC section – Master semester 2 and 4 – 2024-2025

# 1. Limit Equilibrium Method (LEM): problem statement

Applications:

(i) To assess the stability of a given slope (computing the Factor of Safety, *F*).
(ii) To perform the back-analysis of a failed slope to
- study the causes of the collapse;
- compute the operative shear strength parameters during collapse;
- propose and design remedial works.

**Hypothesis 1**: 2D problem.
The slip surface *S* delimits the considered body *B*.

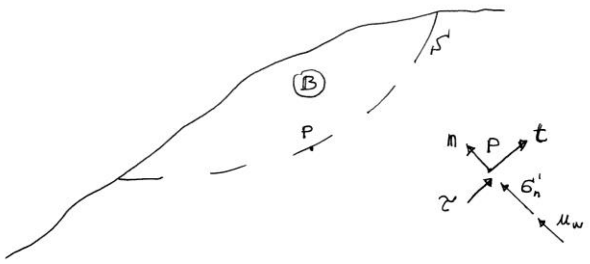

**Figure 1.1.** The limit equilibrium method: outline of stress components at the generic point P of the considered slip surface *S*.

At this stage only fully saturated and dry conditions are considered. At any point P of a given slip surface *S*, **n** and **t** are the unit vectors perpendicular and parallel to the slip surface (**Figure 1.1**). The following stress components are defined in P:

$\sigma_n$, normal total stress;

$t$, shear stress;

$u_w$, pore water pressure;

$\sigma'_n = \sigma_n - u_w$, normal effective stress.

Resultant forces are:

$$\mathbf{N} = \int_S \sigma_n \mathbf{n} dS \tag{1.1}$$

$$\mathbf{T} = \int_S \tau \mathbf{t} dS \tag{1.2}$$

$$\mathbf{U} = \int_S u_w \mathbf{n} dS \tag{1.3}$$

$$\mathbf{N}' = \int_S \sigma'_n \mathbf{n} dS = \int_S (\sigma_n - u_w) \mathbf{n} dS = \mathbf{N} - \mathbf{U} \tag{1.4}$$

**Hypothesis 2:** validity of the Mohr-Coulomb failure criterion.

At each point P, the shear stress at failure (shear strength) is given by

$$\tau_f = c' + \sigma'_n \tan\varphi' \tag{1.5}$$

where $c'$ is the intercept cohesion and $\varphi'$ is the angle of shearing resistance (**Figure 1.2**).
Resultant force from the shear resistance along *S*:

$$\mathbf{T_f} = \int_S \tau_f \mathbf{t} dS \tag{1.6}$$

The body B is in limit equilibrium condition when in every point P of S: $\tau = \tau_f$.

If along *S*, $\tau < \tau_f$, we assume the possibility to bring *B* into a limit equilibrium condition by reducing the available shear strength as follows

$$\tau = \frac{\tau_f}{F} = \frac{c'}{F} + \sigma'_n \frac{\tan\varphi'}{F} \tag{1.7}$$

where *F* is the factor of safety.

The mobilized shear strength parameters are:

$$c_{mob} = \frac{c'}{F} \tag{1.8}$$

$$\varphi'_{mob} = \arctan\frac{\tan\varphi'}{F} \tag{1.9}$$

**Hypothesis 3:** *F* is constant along *S*.

$$\mathbf{T} = \int_S \tau \mathbf{t} dS = \int_S \frac{\tau_f}{F} \mathbf{t} dS = \frac{1}{F}\int_S \tau_f \mathbf{t} dS = \frac{1}{F}\mathbf{T_f} \tag{1.10}$$

A value of *F* can be computed for each *S*; the surface for which *F* attains the minimum value is called the critical slip surface; the corresponding *F* value is the factor of safety of the slope.

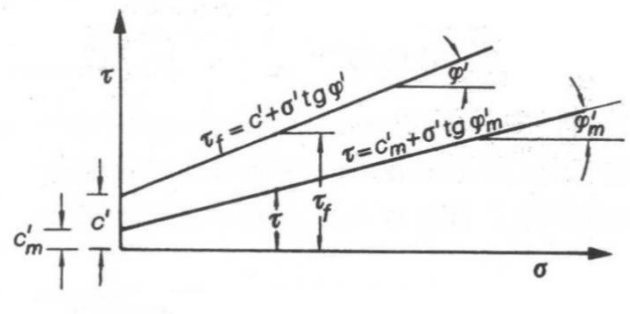


**Figure 1.2.** Available and mobilized shear strength.

**Remarks**

- The LEM does not consider soil displacements prior, during or after slope failure.
- The LEM does not involve any constitutive law for the behaviour of the involved soil.
- The way to compute *F* is different for the various methods.
- *F* < 1 is not an admissible condition.

## 2. Stability analysis of a rigid block

The block in **Figure 2.1** is separated from the stable mass by a vertical discontinuity (BC). In the considered configuration, the discontinuity is filled with water and drainage is allowed at the toe of the block (A).

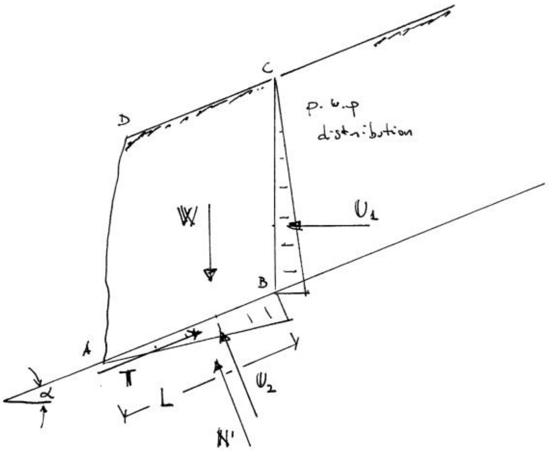


**Figure 2.1.** Stability analysis of a block.

$\mathbf{U}_1$, resultant force of the water pressures acting on the discontinuity;
$\mathbf{U}_2$, resultant force of the water pressures acting on the slip surface.

Equilibrium in the direction parallel to the slip surface:

$$T = U_1 \cos\alpha + W \sin\alpha \tag{2.1}$$

Equilibrium in the direction perpendicular to the slip surface:

$$N' = W \cos\alpha - U_2 - U_1 \sin\alpha \tag{2.2}$$

For a planar slip surface Equation ( 1.10 ) yields

$$T = \frac{T_f}{F} = \int_0^L \frac{\tau_f}{F} dl = \frac{c'L + N' \tan\varphi'}{F} \tag{2.3}$$

Factor of safety:

$$F = \frac{T_f}{T} = \frac{c'L + \left(W \cos\alpha - U_2 - U_1 \sin\alpha\right)\tan\varphi'}{W \sin\alpha + U_1 \cos\alpha} \tag{2.4}$$

**Remarks**

- For planar slip surfaces, $F$ is computed considering two equations of static equilibrium in the perpendicular and parallel directions to the slip surface.
- Rotational equilibrium equation can be used to compute the position of the force $\mathbf{N}'$.
- Water pressures reduce the resultant available shear strength ($T_f$) and increase the mobilized shear strength ($T$).
- The distribution of $\sigma'_n$ does not affect the value of $F$.
- Different distributions of the water pressure may apply depending on the configuration (geometry of the block and discontinuities) and drainage conditions.

## 3. Infinite slope analysis

Application: extensive slips involving relatively shallow soil covers over much more competent materials. The soil cover tends to slide on a plane parallel to the soil surface.

The stability of a slice is considered.

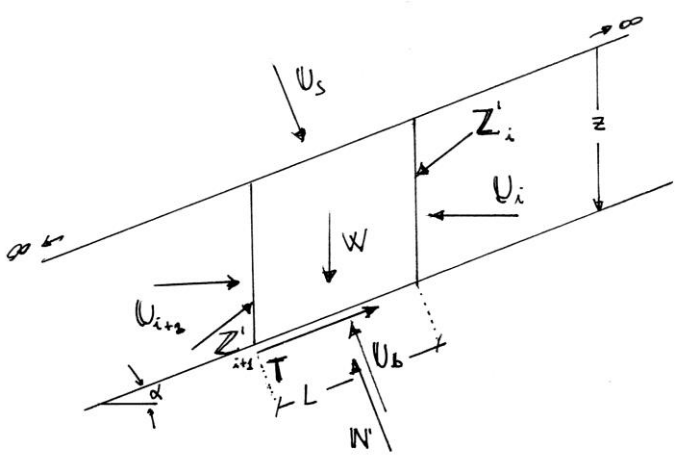


**Figure 3.1.** Infinite slope analysis.

$\mathbf{U}_i$ (resp. $\mathbf{U}_{i+1}$) resultant force of the water pressures acting on the right (resp. left) boundary of the slice. $\mathbf{U}_b$ (resp. $\mathbf{U}_S$) resultant force of the water pressures acting on the bottom (resp. top) boundary of the slice.
$\mathbf{Z}'_i$ (resp. $\mathbf{Z}'_{i+1}$) resultant force of the effective stresses acting on the right (resp. left) boundary of the slice. For symmetry: $\mathbf{Z}'_i + \mathbf{Z}'_{i+1} = \mathbf{0}$.

Equilibrium in the direction parallel to the slip surface:

$$T = W \sin\alpha + (U_i - U_{i+1})\cos\alpha \qquad (3.1)$$

Equilibrium in the direction perpendicular to the slip surface:

$$N' = W\cos\alpha - (U_i - U_{i+1})\sin\alpha - (U_b - U_s) \qquad (3.2)$$

For a planar slip surface Equation ( 1.10 ) yields

$$T = \frac{T_f}{F} = \int_0^L \frac{\tau_f}{F} dl = \frac{c'L + N'\tan\varphi'}{F} \qquad (3.3)$$

Factor of safety:

$$F = \frac{T_f}{T} = \frac{c'L + \left(W\cos\alpha - (U_i - U_{i+1})\sin\alpha - (U_b - U_s)\right)\tan\varphi'}{W\sin\alpha + (U_i - U_{i+1})\cos\alpha} \qquad (3.4)$$

For dry conditions

$$F = \frac{c'L}{W \sin \alpha} + \frac{\tan \varphi'}{\tan \alpha} = \frac{c'}{\gamma z \sin \alpha \cos \alpha} + \frac{\tan \varphi'}{\tan \alpha} \quad (3.5)$$

where $\gamma$ is the unit weight of the soil.

The critical depth $z_{cr}$ at which $F$=1:

$$z_{cr} = \frac{c'}{\gamma \cos \alpha \,(\sin \alpha - \cos \alpha \tan \phi')\,)} \quad (3.6)$$

For hydrostatic conditions

$$F = \frac{c'}{\gamma' z \sin \alpha \cos \alpha} + \frac{\tan \varphi'}{\tan \alpha} \quad (3.7)$$

where $\gamma'$ is the buoyant (submerged) unit weight of the soil.

For seepage parallel to slope

$$F = \frac{c'}{\gamma_{sat} z \sin \alpha \cos \alpha} + \frac{\gamma'}{\gamma_{sat}} \frac{\tan \varphi'}{\tan \alpha} \quad (3.8)$$

where $\gamma_{sat}$ is the unit weight of the soil in saturated conditions.

**Remarks**

- Flow condition has a strong effect on the stability of the slope: Equation (3.8) vs Equation (3.7).
- Alternate method for considering water in the stability analysis is to place in equilibrium: (i) the buoyant weight of the slice, (ii) the resultant of the effective stress distributions acting along the boundary of the slice, (iii) the resultant of the shear stress distributions acting along the boundary of the slice, (iv) the seepage force equals the product of volume of the considered element (where the water flows) times the unit weight of water times the hydraulic gradient ($i$). The seepage force acts parallel to the flow lines. For seepage parallel to slope $i = \sin \alpha$.

## 4. LEM for circular slip surfaces

The resultant forces in the case of a circular slip surface are depicted in **Figure 4.1**. It is shown that the computation of $F$ is statically indeterminate for circular slip surfaces.

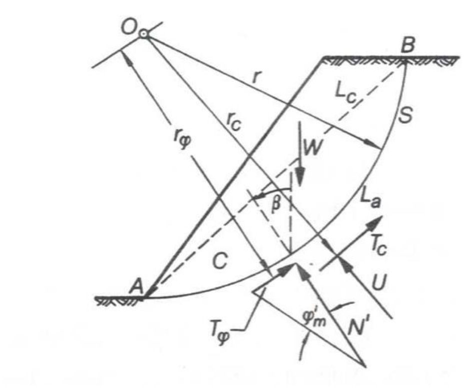


**Figure 4.1.** The limit equilibrium method applied for a circular slip surface (Airò Farulla, 2001).

Known forces acting on the body $B$ are **W** and **U**. The force $\mathbf{N}'$ passes through the center O as every vector $\sigma'_n \mathbf{n}$ does. The inclination ($\beta$) and the magnitude $N'$ are unknown.

It is convenient to write $\mathbf{T}$ as:

$$\mathbf{T} = \mathbf{T}_c + \mathbf{T}_\varphi \quad (4.1)$$

with

$$\mathbf{T}_c = \int_{L_a} \frac{c'}{F} \mathbf{t} dL_a \quad (4.2)$$

$$\mathbf{T}_\varphi = \int_{L_a} \frac{\sigma'_n \tan\varphi'}{F} \mathbf{t} dL_a \quad (4.3)$$

where $L_a$ is the length of the circular arc AB.

The magnitude of $\mathbf{T}_c$ is

$$T_c = \frac{c' L_c}{F} \quad (4.4)$$

where $L_c$ is the length of the chord AB. The direction of $\mathbf{T}_c$ is parallel to the chord AB and has a distance from point O equal to:

$$r_c = r \frac{L_a}{L_c} \quad (4.5)$$

The magnitude of $\mathbf{T}_\varphi$ is

$$T_\varphi = \frac{N' \tan\varphi'}{F} \quad (4.6)$$

The direction of $\mathbf{T}_c$ is perpendicular to $\mathbf{N}'$ and its distance ($r_\phi$) from point O is unknown.

Finally four unknowns are needed ($N'$, $\beta$, $r_\phi$, $F$) while 3 equilibrium equations are available in a 2D problem (two equations of equilibrium in two orthogonal directions + one equation of rotational equilibrium).

**Remarks**

- The undetermined computation of $F$ would be solved if the distribution of normal effective stresses ($\sigma'_n$) along $L_a$ would be known, for example through a stress:strain analysis. However the LME does not make any assumption on the constitutive behaviour of the involved materials.
- In the Friction Circle Method it is assumed (arbitrarily) that $r_\phi = r_c$. The computation of $F$ becomes then possible using the three equilibrium conditions.

## 5. The method of slices

The method of slices is an application of the LEM. It can be used to compute $F$ for an arbitrary shape of the slip surface $S$. The body $B$ can be composed of different soils with different mechanical properties. In the method of slices, the body B is divided into $n$ slices by $n$-1 (conventionally) vertical lines. Forces acting on a single slice are depicted in **Figure 5.1**:

$\mathbf{W}_i$, force weight of the slice $i$;

$\mathbf{U}_{bi}$, resultant force of the pore water pressures acting on the bottom of the slice $i$;

$\mathbf{U}_i$ and $\mathbf{U}_{i-1}$, resultant forces of the pore water pressures acting on the left and right sides of the slice $i$;

$\mathbf{E}'_i$ and $\mathbf{E}'_{i-1}$, resultant forces of the effective stresses normal to the left and right sides of the slice $i$; these forces act at a distance $h_i$ and $h_{i-1}$ from the bottom of the slice.

$\mathbf{X}_i$ and $\mathbf{X}_{i-1}$, resultant forces of the shear stresses acting on the left and right sides of the slice $i$;

$\mathbf{N}'_i$, resultant forces of the effective stresses normal to the bottom of the slice $i$; if the slice is sufficiently small in width, $\mathbf{N}'_i$ can be considered as applied at the center of the bottom of the slice;

$\mathbf{T}_i$, resultant of the shear stress acting on the bottom of the slice $i$; for an appropriate subdivision in slices of the body B, the arc at the bottom of the slice can be approximated with the chord; in this case the direction and position of $\mathbf{T}_i$ is known, since the force is applied directly along the base of the slice.

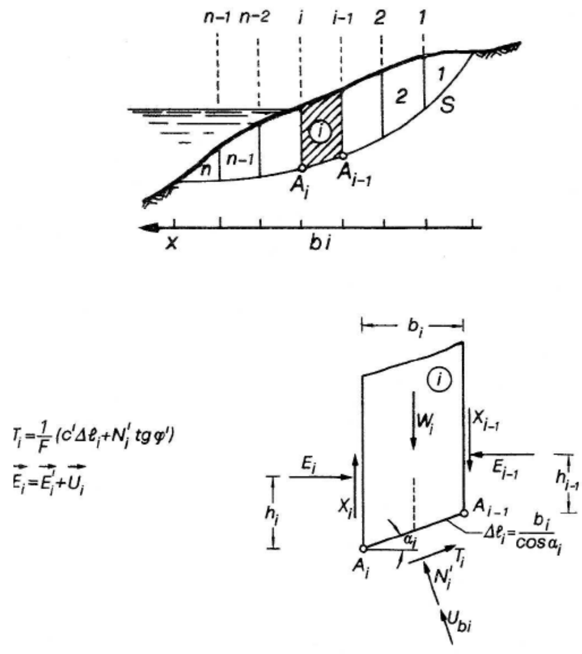


**Figure 5.1.** The method of slices (Airò Farulla, 2001).

A total of 4$n$ equations are available for computing $F$:

- three equations of equilibrium for each slice: 3 x $n$;
- relationship between the magnitudes of $\mathbf{N}'_i$ and $\mathbf{T}_i$ through $F$: $n$.

There are 5$n$-2 unknowns for the computation of $F$:

- magnitude and position of the inter-slice forces ($E'_i$ , $h_i$, $X_i$): 3 x ($n$-1);
- magnitude of the forces acting on the bottom of the slices ($N'_i$ o $T_i$ ,): 2 x $n$;
- the factor of safety (F): 1.

A total of $n$-2 additional equations are needed to compute $F$. Several methods exist which propose different choices of the additional conditions to compute $F$. The methods can be distinguished in

- rigorous: the additional $n$-2 conditions are selected in order to respect all the 3$n$ equations of equilibrium;
- non-rigorous: as a result of the additional conditions (e.g. some forces are neglected or their direction is fixed arbitrarily) some of the equations of equilibrium are not used neither verified.

## 6. The method of slices for circular slip surfaces

When the slip surface is circular, it is convenient to compute *F* considering the rotational equilibrium equation of the entire body *B* with respect to the center O. As depicted in **Figure 6.1**, the moments of forces $\mathbf{U}_{bi}$ and $\mathbf{N}'_i$, are equal to zero. The moments of forces $\mathbf{E}'_i$ and $\mathbf{X}_i$ are not considered since those are internal forces for the body *B*. From the rotational equilibrium *F* is computed as:

$$F = \frac{\sum T_i r}{\sum W_i r \sin \alpha_i} = \frac{\sum (c' \Delta l_i + N'_i \tan \varphi')}{\sum W_i \sin \alpha_i} \quad (6.1)$$

When using this equation, unknown terms exist only in the numerator of the fraction. On the other hand, unknowns would appear on the numerator and denominator if *F* would be computed through the equation of equilibrium in the vertical or horizontal direction; in this sense, the effects of the assumed additional conditions on the computation of *F* is less significant when the above equation is used for a circular slip surface.

The most common non-rigorous methods, which use Equation 6.1, are the Fellenius' method the Bishop's simplified method, which are discussed in the next sections.

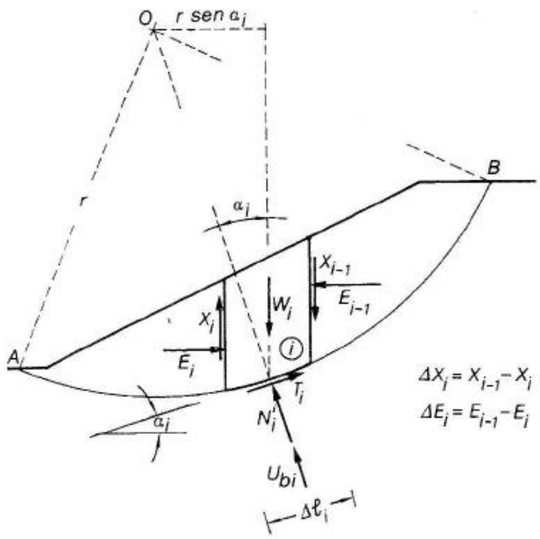


**Figure 6.1.** The method of slices for a circular slip surface (Airò Farulla, 2001).

## 7. The Fellenius' method

The Fellenius' method is also called the ordinary method of slices or the Swedish method. In order to compute $N_i'$ the equilibrium of each slice is considered in the direction perpendicular to base of the slice:

$$N_i' = W_i \cos\alpha_i - U_{bi} - \left(X_i - X_{i-1}\right)\cos\alpha_i - \left(E_{i-1} - E_i\right)\sin\alpha_i \quad (7.1)$$

The additional conditions regard the inclination of the inter-slice forces, which are assumed parallel to the base of the slice for each slice:

$$N_i' = W_i \cos\alpha_i - U_{bi} \quad (7.2)$$

Equations 6.1 writes

$$F = \frac{\sum\left[c'\Delta l_i + \left(W_i \cos\alpha_i - U_{bi}\right)\tan\varphi'\right]}{\sum W_i \sin\alpha_i} \quad (7.3)$$

**Remarks**

- The computation of $F$ is straightforward since it uses a linear equation.
- The method has been proven to be poorly accurate, especially for deep slip surfaces.

## 8. The Bishop's simplified method

Bishop's simplified method is a non-rigorous method of slices for circular slip surfaces. The method uses the equation of rotational equilibrium of the entire body *B* (Equation 6.1) to compute *F*.

The magnitude of the resultant force of the effective stresses normal to the base of each slice ($N_i'$) is computed considering the equilibrium of the slice in the vertical direction:

$$N_i' = \frac{W_i - \left(X_i - X_{i-1}\right) - U_{bi}\cos\alpha_i - \dfrac{c'\Delta l_i \sin\alpha_i}{F}}{\cos\alpha_i + \dfrac{\sin\alpha_i \tan\varphi'}{F}} \tag{8.1}$$

or

$$N_i' = \frac{W_i - \left(X_i - X_{i-1}\right) - U_{bi}\cos\alpha - \dfrac{c'\Delta l_i \sin\alpha_i}{F}}{m_{\alpha_i}} \tag{8.2}$$

with

$$m_{\alpha_i} = \cos\alpha_i + \frac{\sin\alpha_i \tan\varphi'}{F} \tag{8.3}$$

**Hypothesis:** It is assumed that for each slice the tangential inter-slice forces have equal magnitude and opposite direction $\left(X_i - X_{i-1}\right) = 0$.

Equation 5.1 becomes

$$F = \frac{\sum\left[c'b_i + \left(W_i - U_{bi}\cos\alpha_i\right)\tan\varphi'\right]\dfrac{1}{m_{\alpha_i}}}{\sum W_i \sin\alpha_i} \tag{8.4}$$

**Remarks**

- An iterative procedure is required in order to compute *F*.
- For circular slip surfaces, the Bishop's simplified method provides values of *F* similar to the ones computed with rigorous methods.

## 9. The General Limit Equilibrium Method

Several rigorous LEMs have been proposed over the last decades. They all introduce assumptions so that the computation of F is carried out ensuring, at the same time, the respect of all the equilibrium equations for all the slices. In this course only the GLE method (Fredlund and Kranhn, 1977) will be introduced. Several rigorous and non-rigorous methods can be seen as particular applications of the GLE method.

**Figure 9.1** depicts a non-circular slip surface. Along with the forces introduced in the previous sections, the following actions are also considered:
$\mathbf{A}_r$ (resp. $\mathbf{A}_s$) resultant force of the water pressures acting on the right (resp. left) vertical boundary of the landslide body;
$\mathbf{Q}_i$ (resp. $\mathbf{P}_i$) horizontal (resp. vertical) force acting on the surface of the slice *i*;
$k\mathbf{W}_i$, horizontal mass force acting on the slice *i*.

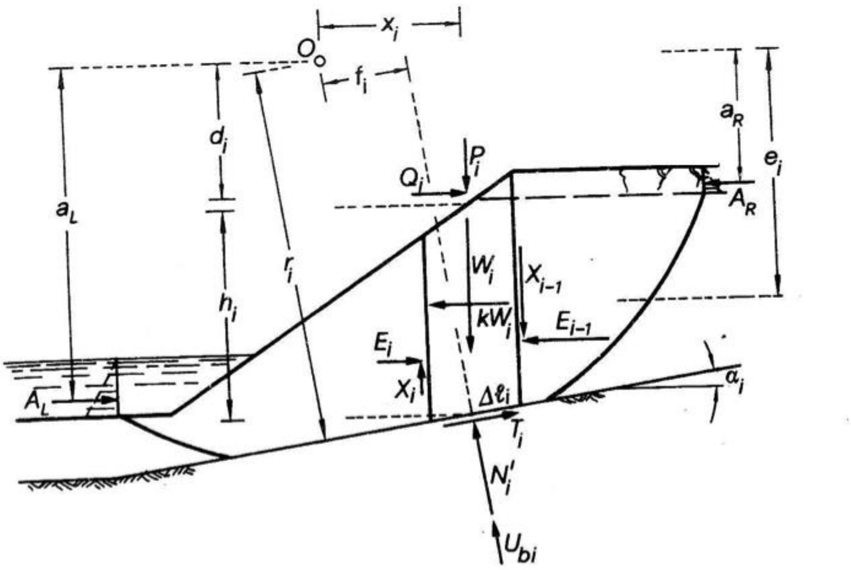


**Figure 9.1.** The GLE method (Airò Farulla, 2001).

The magnitude of $N_i'$ is computed considering the equilibrium of the slice in the vertical direction:

$$N_i' = \frac{W_i + \left(X_{i-1} - X_i\right) + P_i - U_{bi}\cos\alpha_i - \dfrac{c'\Delta l_i \sin\alpha_i}{F}}{\cos\alpha_i + \dfrac{\sin\alpha_i \tan\varphi'}{F}} \tag{9.1}$$

Equilibrium in the horizontal direction for the slice *i*:

$$\left(E_{i-1} - E_i\right) + kW_i - Q_i + \left(N' + U_{bi}\right)\sin\alpha_i = \frac{c'\Delta l_i + N_i'\tan\varphi'}{F}\cos\alpha_i \tag{9.2}$$

Equilibrium in the horizontal direction for the entire body gives:

$$F_f = \frac{\sum\left(c'\Delta l_i + N_i'\tan\varphi'\right)\cos\alpha_i}{\sum\left(N_i' + U_{bi}\right)\sin\alpha_i + \sum kW_i - \sum Q_i + \left(A_r - A_l\right)} \tag{9.3}$$

where $F$ is named $F_f$ since the equilibrium of forces has been used.

The rotational equilibrium with respect to an arbitrary point O gives the following expression for the factor of safety:

$$F_m = \frac{\sum\left(c'\Delta l_i + N_i' \tan\varphi'\right)r_i}{\sum W_i x_i + \sum kW_i e_i + \sum P_i x_i - \sum Q_i d_i + \left(A_r a_r - A_l a_l\right) - \sum\left(N' + U_{bi}\right)f_i} \qquad (9.4)$$

The last two equations have $n$ unknowns ($F$, $X_i$). The problem is solved introducing $n$-1 equations of the type:

$$X(x) = E(x)\lambda f(x) \qquad (9.5)$$

in which the inclinations of the inter-slice forces are fixed through a selected function $f(x)$ and an additional unknown $\lambda$.

Several LEMs can be seen as particular applications of the GLE methods; in particular the Bishop's simplified method can be obtained imposing $X_i$=0 and $\lambda$=0. Of large application is the Morgenstern and Price method (Morgenstern and Price, 1965), which uses Equation (9.5) for a division of the body in slices with an infinitesimal width.

**Remarks**

- An iterative procedure is required in order to compute $F$.
- The computations of $F_f$ and $F_m$ allow to highlight the advantage of using the rotational equilibrium in the method of slices.